Descent Numbers and Major Indices for the Hyperoctahedral Group

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December 1, 2000

Dedicated to Dominique Foata

[1] Research of all authors was supported in part by the Israel Science Foundation, administered by the Israel Academy of Sciences and Humanities.

Abstract

We introduce and study three new statistics on the hyperoctahedral group B_n, and show that they give two generalizations of Carlitz's identity for the descent number and major index over S_n. This answers a question posed by Foata.

1 Introduction

A well known classical result due to MacMahon (see [13]) asserts that the inversion number and major index of a permutation are equidistributed over the symmetric group S_n. The joint distribution of major index and descent number was studied by Carlitz [4], Gessel [11], and others. Despite the fact that an increasing number of enumerative results of this nature have been generalized to the hyperoctahedral group B_n (see, e.g., [3], [9], [16], [17], [18]) and that several "major index" statistics have been introduced and studied for B_n (see, e.g., [5], [6], [7], [14], [15], [20]), no generalization of MacMahon's result has been found until the recent paper [1]. There a new statistic, the *flag major index* (denoted here *fmaj*) was introduced; and it was shown to be equidistributed with length, which is the natural analogue of inversion number from a Coxeter group theoretic point of view. No corresponding "descent statistic" has been found that allows the generalization to B_n of the well known Carlitz identity for descent number and major index on S_n (see [4], and also Theorem 2.2 below), a problem first posed by Foata [8].

Problem 1.1 *(Foata) : Extend the ("Euler-Mahonian") bivariate distribution of descent number and major index to the hyperoctahedral group* B_n*.*

The purpose of this paper is to introduce and study three new statistics on B_n: the *negative descent* (denoted *ndes*); the *negative major* (*nmaj*); and the *flag descent* (*fdes*). When restricted to S_n they reduce to descent number, major index, and twice descent number, respectively, and they solve the above problem. More precisely, we show that *nmaj* is equidistributed with length on B_n, and that *ndes* is the "right" corresponding descent statistic needed to extend Carlitz's result to B_n, thus answering Foata's question. Finally, we prove the surprising result that the pair of statistics (*fdes*, *fmaj*) is equidistributed with (*ndes*, *nmaj*) over B_n, thus obtaining a second generalization of Carlitz's identity to B_n.

The organization of the paper is as follows. In the next section we collect several definitions, notation, and results that will be used in the sequel. In section 3 we introduce a new "descent set" for elements of B_n and, correspondingly, a new "descent number" *ndes* and a new "major index" *nmaj*. It is shown that *nmaj* is equidistributed with length over B_n (Proposition 3.1), and, moreover, that the pair $(ndes, nmaj)$ gives a generalization of Carlitz's identity (Theorem 3.2) and thus solves Foata's problem. In section 4 we introduce a "flag analogue" of the descent number for B_n, *fdes*, and show that it gives a second solution to Foata's problem (Corollary 4.2). We then deduce that the pairs of statistics $(ndes, nmaj)$ and $(fdes, fmaj)$ are equidistributed over B_n (Corollary 4.5).

2 Notation, Definitions, and Preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this paper. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the ring of integers, and $\mathbf{Q}$ be the field of rational numbers; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, \ldots, a\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). Given $n, m \in \mathbf{Z}$, $n \leq m$, we let $[n, m] \stackrel{\text{def}}{=} \{n, n+1, \ldots, m\}$. For $S \subset \mathbf{N}$ we write $S = \{a_1, \ldots, a_r\}_<$ to mean that $S = \{a_1, \ldots, a_r\}$ and $a_1 < \ldots < a_r$. The cardinality of a set A will be denoted by $|A|$. More generally, given a multiset $M = \{1^{a_1}, 2^{a_2}, \ldots, r^{a_r}\}$ we denote by $|M|$ its cardinality, so $|M| = \sum_{i=1}^{r} a_i$. Given a statement P we will sometimes find it convenient to let

$$\chi(P) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } P \text{ is true,} \\ 0, & \text{if } P \text{ is false.} \end{cases}$$

Given a variable q and a commutative ring R we denote by $R[q]$ (respectively, $R[[q]]$) the ring of polynomials (respectively, formal power series) in q with coefficients in R. For $i \in \mathbf{N}$ we let, as customary, $[i]_q \stackrel{\text{def}}{=} 1 + q + q^2 + \ldots + q^{i-1}$ (so $[0]_q = 0$).

Given a sequence $\sigma = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ we say that a pair $(i, j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $a_i > a_j$. We say that $i \in [n-1]$ is a *descent* of σ if $a_i > a_{i+1}$. We denote by $inv(\sigma)$ (respectively, $des(\sigma)$) the number of inversions (respectively, descents) of σ. We also let

$$maj(\sigma) \stackrel{\text{def}}{=} \sum_{\{i:\, a_i > a_{i+1}\}} i$$

and call it the *major index* of σ.

Given a set T we let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n \stackrel{\rm def}{=} S([n])$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1 \ldots \sigma_n$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$. If $\sigma \in S_n$ then we will also write σ in *disjoint cycle form* (see, e.g., [19], p. 17), and will usually omit to write the 1-cycles of σ. For example, if $\sigma = 365492187$ then we also write $\sigma = (9, 7, 1, 3, 5)(2, 6)$. Given $\sigma, \tau \in S_n$ we let $\sigma\tau \stackrel{\rm def}{=} \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$.

We denote by B_n the group of all bijections π of the set $[-n, n] \setminus \{0\}$ onto itself such that

$$\pi(-a) = -\pi(a)$$

for all $a \in [-n, n] \setminus \{0\}$, with composition as the group operation. This group is usually known as the group of "signed permutations" on $[n]$, or as the *hyperoctahedral group* of rank n. We identify S_n as a subgroup of B_n, and B_n as a subgroup of S_{2n}, in the natural ways.

If $\pi \in B_n$ then we write $\pi = [a_1, \ldots, a_n]$ to mean that $\pi(i) = a_i$ for $i = 1, \ldots, n$, and we let

$$\begin{aligned} inv(\pi) &\stackrel{\rm def}{=} inv(a_1, \ldots, a_n), \\ des_A(\pi) &\stackrel{\rm def}{=} des(a_1, \ldots, a_n), \\ maj_A(\pi) &\stackrel{\rm def}{=} maj(a_1, \ldots, a_n), \\ Neg(\pi) &\stackrel{\rm def}{=} \{i \in [n] : \ a_i < 0\}, \end{aligned} \tag{1}$$

and

$$neg(\pi) \stackrel{\rm def}{=} |Neg(\pi)|.$$

It is well known (see, e.g., [2, Proposition 8.1.3]) that B_n is a Coxeter group with respect to the generating set $\{s_0, s_1, s_2, \ldots, s_{n-1}\}$, where

$$s_0 \stackrel{\rm def}{=} [-1, 2, \ldots n]$$

and

$$s_i \stackrel{\rm def}{=} [1, 2, \ldots, i-1, i+1, i, i+2, \ldots n]$$

for $i = 1, \ldots, n-1$. This gives rise to two other natural statistics on B_n (similarly definable for any Coxeter group), namely

$$l(\pi) \stackrel{\rm def}{=} min\{r \in \mathbf{N} : \ \pi = s_{i_1} \ldots s_{i_r} \quad \text{for some} \quad i_1, \ldots, i_r \in [0, n-1]\}$$

(known as the *length* of π) and

$$des_B(\pi) \stackrel{\text{def}}{=} |\{i \in [0, n-1] : l(\pi s_i) < l(\pi)\}|.$$

There is a well known direct combinatorial way to compute these two statistics for $\pi \in B_n$ (see, e.g., [2, Propositions 8.1.1 and 8.1.2] or [3, Proposition 3.1 and Corollary 3.2]), namely

$$l(\pi) = inv(\pi) - \sum_{i \in Neg(\pi)} \pi(i) \tag{2}$$

and

$$des_B(\pi) = |\{i \in [0, n-1] :\ \pi(i) > \pi(i+1)\}|, \tag{3}$$

where $\pi(0) \stackrel{\text{def}}{=} 0$. For example, if $\pi = [-3, 1, -6, 2, -4, -5] \in B_6$ then $inv(\pi) = 9$, $des_A(\pi) = 3$, $maj_A(\pi) = 11$, $neg(\pi) = 4$, $l(\pi) = 27$, and $des_B(\pi) = 4$.

Let

$$T \stackrel{\text{def}}{=} \{\pi \in B_n :\ des_A(\pi) = 0\}. \tag{4}$$

It is then well known, and easy to see, that

$$B_n = \biguplus_{u \in S_n} \{\pi u :\ \pi \in T\}, \tag{5}$$

where $\uplus$ denotes disjoint union.

We will use this decomposition often in this paper. The reader familiar with Coxeter groups will immediately recognize that (5) is one case of the multiplicative decomposition of a Coxeter group into a parabolic subgroup and its minimal coset representatives (see, e.g., [12] or [2]).

As customary, given a variable t we define an operator $\delta_t :\ \mathbf{Q}[q, t] \to \mathbf{Q}[q, t]$ by

$$\delta_t(P(q, t)) = \frac{P(q, qt) - P(q, t)}{qt - t},$$

for all $P \in \mathbf{Q}[q, t]$. Note that $\delta_t(q^n) = 0$ and

$$\delta_t(t^n) = [n]_q\, t^{n-1} \tag{6}$$

for all $n \in \mathbf{N}$, and

$$\delta_t(A(q, t)\, B(q, t)) = \delta_t(A(q, t))B(q, t) + A(q, qt)\delta_t(B(q, t)) \tag{7}$$

for all $A, B \in \mathbf{Q}[q, t]$. Also,

$$\delta_t(P)(1, t) = \frac{d}{dt}(P(1, t))$$

for all $P \in \mathbf{Q}[q,t]$.

For $n \in \mathbf{P}$ we let

$$A_n(t,q) \stackrel{\mathrm{def}}{=} \sum_{\sigma \in S_n} t^{des_A(\sigma)} q^{maj_A(\sigma)},$$

and $A_0(t,q) \stackrel{\mathrm{def}}{=} 1$. For example, $A_3(t,q) = 1 + 2tq^2 + 2tq + t^2q^3$. The following two results are due to Carlitz [4] and Gessel [11], and proofs of them can also be found in [10].

Theorem 2.1 *Let $n \in \mathbf{P}$. Then*

$$A_n(t,q) = (1 + tq[n-1]_q)\, A_{n-1}(t,q) + tq(1-t)\delta_t(A_{n-1}(t,q)).$$

Theorem 2.2 *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{A_n(t,q)}{\prod_{i=0}^{n} (1 - tq^i)}.$$

in $\mathbf{Z}[q][[t]]$.

3 The "Negative" Statistics

In this section we define and study a new "descent set" for the elements of B_n. This gives rise, in a very natural way, to new "major index" and "descent number" statistics for B_n. We then show that these two statistics give a generalization of Carlitz's identity to B_n, and that one of them is equidistributed with length.

3.1 The Negative Descent Multiset

For $\pi \in B_n$ we define

$$Des_A(\pi) \stackrel{\mathrm{def}}{=} \{i \in [n-1] : \ \pi(i) > \pi(i+1)\}$$

and the *negative descent multiset*

$$NDes(\pi) \stackrel{\mathrm{def}}{=} Des_A(\pi) \biguplus \{-\pi(i) : \ i \in Neg(\pi)\}, \tag{8}$$

where $Neg(\pi)$ is the set of positions of negative entries in π, defined in (1).

For example, if $\pi = [-3, 1, -6, 2, -4, -5] \in B_6$ then $Des_A(\pi) = \{2, 4, 5\}$ and $NDes(\pi) = \{2, 3, 4^2, 5^2, 6\}$. Note that if $\pi \in S_n$ then $NDes(\pi)$ is a set and coincides with the usual descent set of π. Also note that $NDes(\pi)$ can be defined rather naturally also in purely Coxeter group theoretic terms. In fact, for $i \in [n]$ let $\eta_i \in B_n$ be defined by

$$\eta_i \stackrel{\text{def}}{=} [1, 2, \ldots, i-1, -i, i+1, \ldots, n],$$

so $\eta_1 = s_0$. Then $\eta_1, \ldots, \eta_n$ are reflections (in the Coxeter group sense; see, e.g., [12]) of B_n and it is clear from (2) that

$$NDes(\pi) = \{i \in [n-1] : \ l(\pi s_i) < l(\pi)\} \biguplus \{i \in [n] : \ l(\pi^{-1}\eta_i) < l(\pi^{-1})\}.$$

These considerations explain why it is natural to think of $NDes(\pi)$ as a "descent set". With this in mind, the following definitions are also natural. For $\pi \in B_n$ we let

$$ndes(\pi) \stackrel{\text{def}}{=} |NDes(\pi)|$$

and

$$nmaj(\pi) \stackrel{\text{def}}{=} \sum_{i \in NDes(\pi)} i.$$

For example, if $\pi = [-3, 1, -6, 2, -4, -5] \in B_6$ then $ndes(\pi) = 7$ and $nmaj(\pi) = 29$.

Note that from (8) it follows that

$$nmaj(\pi) = maj_A(\pi) - \sum_{i \in Neg(\pi)} \pi(i) \qquad (\forall \pi \in B_n). \tag{9}$$

This formula is also one of the motivations behind our definition of $nmaj(\pi)$, because of the corresponding formula (2).

3.2 Equidistribution and Generating Function

Our first result shows that $nmaj$ and l are equidistributed in B_n.

Proposition 3.1 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\pi \in B_n} q^{nmaj(\pi)} = \sum_{\pi \in B_n} q^{l(\pi)}.$$

Proof. Let T be defined by (4). It is clear from our definitions that for all $u \in S_n$ and $\sigma \in T$,

$$maj_A(\sigma u) = maj_A(u), \qquad inv(\sigma u) = inv(u),$$

and

$$\sum_{i \in Neg(\sigma u)} (\sigma u)(i) = \sum_{i \in Neg(\sigma)} \sigma(i).$$

Therefore, from (2), (5), (9), and the corresponding classical result for S_n (see, e.g., [13]) we conclude that

$$\begin{aligned}
\sum_{\pi \in B_n} q^{nmaj(\pi)} &= \sum_{\sigma \in T} \sum_{u \in S_n} q^{nmaj(\sigma u)} \\
&= \sum_{\sigma \in T} \sum_{u \in S_n} q^{maj_A(\sigma u) - \sum_{i \in Neg(\sigma u)} (\sigma u)(i)} \\
&= \sum_{\sigma \in T} q^{-\sum_{i \in Neg(\sigma)} \sigma(i)} \cdot \sum_{u \in S_n} q^{maj_A(u)} \\
&= \sum_{\sigma \in T} q^{-\sum_{i \in Neg(\sigma)} \sigma(i)} \cdot \sum_{u \in S_n} q^{inv(u)} \\
&= \sum_{\sigma \in T} \sum_{u \in S_n} q^{inv(\sigma u) - \sum_{i \in Neg(\sigma u)} (\sigma u)(i)} \\
&= \sum_{\pi \in B_n} q^{l(\pi)},
\end{aligned}$$

as desired. $\square$

We now prove the main result of this section, which is also the first main result of this work, namely that the pair of statistics $(ndes, nmaj)$ gives a generalization of Theorem 2.2 to B_n.

Theorem 3.2 *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\pi \in B_n} t^{ndes(\pi)} q^{nmaj(\pi)}}{(1-t) \prod_{i=1}^{n} (1 - t^2 q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

Proof. Let T have the same meaning as in (4). Then it is clear from our definitions that $des_A(\sigma u) = des_A(u)$ for all $\sigma \in T$ and $u \in S_n$. Therefore we have from (5) that

$$\sum_{\pi \in B_n} t^{ndes(\pi)} q^{nmaj(\pi)} = \sum_{\sigma \in T} \sum_{u \in S_n} t^{des_A(\sigma u) + neg(\sigma u)} q^{maj_A(\sigma u) - \sum_{i \in Neg(\sigma u)} (\sigma u)(i)}$$

$$
\begin{aligned}
&= \sum_{\sigma\in T} t^{neg(\sigma)} q^{-\sum_{i\in Neg(\sigma)}\sigma(i)} \cdot \sum_{u\in S_n} t^{des_A(u)} q^{maj_A(u)} \\
&= \sum_{S\subseteq[n]} t^{|S|}\, q^{\sum_{i\in S} i} \cdot \sum_{u\in S_n} t^{des_A(u)} q^{maj_A(u)} \\
&= \prod_{i=1}^{n}(1+tq^i) \cdot \sum_{u\in S_n} t^{des_A(u)}\, q^{maj_A(u)}
\end{aligned}
$$

and the result follows from Theorem 2.2. □

4 The Flag Statistics

In this section we introduce yet another pair of statistics on B_n, and show that it also gives a solution to Foata's problem. We then derive some consequences of this result.

4.1 The Flag Major Index

For $i = 0, 1, \ldots, n-1$ let $t_i \stackrel{\text{def}}{=} s_i s_{i-1} \cdots s_0$. Explicitly,

$$t_i = [-i-1, 1, 2, \ldots, i, i+2, \ldots, n]$$

for $i = 0, \ldots, n-1$. It is not hard to show (see also [1]) that for any $\pi \in B_n$ there exist unique integers $k_0, \ldots, k_{n-1}$, with $0 \le k_i \le 2i+1$ for $i = 0, \ldots, n-1$, such that

$$\pi = t_{n-1}^{k_{n-1}} \ldots t_2^{k_2} t_1^{k_1} t_0^{k_0}.$$

The *flag major index* of π (see [1]) is then defined by

$$fmaj(\pi) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} k_i.$$

There is a simple way to compute the flag major index of a signed permutation π. In fact, we have the following result which was first proved in [1].

Theorem 4.1 *Let $\pi \in B_n$. Then*

$$fmaj(\pi) = 2\, maj_A(\pi) + neg(\pi).$$

For example, if $\pi = [-3, 1, -6, 2, -4, -5]$ then $fmaj(\pi) = 2 \cdot 11 + 4 = 26$.

In [1] it was shown that *fmaj* appears naturally in the Hilbert series of (diagonal action) invariant algebras; note that this property is not shared by *nmaj*.

4.2 The Flag Descent Number

For $\pi \in B_n$ let

$$fdes(\pi) \stackrel{\text{def}}{=} 2\, des_A(\pi) + \varepsilon_1(\pi), \tag{10}$$

where des_A is as in Section 2 and

$$\varepsilon_1(\pi) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \pi(1) < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

This definition is motivated by Theorem 4.1, and because of this similarity we call $fdes(\pi)$ the *flag descent number* of π. For example, if $\pi = [-3, 1, -6, 2, -4, -5]$ then $fdes(\pi) = 2 \cdot 3 + 1 = 7$. Note that from (3), (1) and (11) it follows immediately that

$$fdes(\pi) = des_A(\pi) + des_B(\pi) \tag{12}$$

and also

$$fdes(\pi) = des(\pi(-n), \ldots, \pi(-1), \pi(1), \ldots, \pi(n))$$

for all $\pi \in B_n$.

Our aim is to show that the pair of statistics $(fdes, fmaj)$ gives a solution to Foata's problem and thus has the same joint distribution, over B_n, as the pair $(ndes, nmaj)$ defined in the previous section.

4.3 Main Theorems

The main result of this section is

Theorem 4.2 *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\pi \in B_n} t^{fdes(\pi)} q^{fmaj(\pi)}}{(1-t)\prod_{i=1}^{n}(1 - t^2 q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

This solves Foata's problem and implies, together with Theorem 3.2, that the two pairs of statistics $(fdes, fmaj)$ and $(ndes, nmaj)$ are equidistributed over B_n.

Theorem 4.2 will be proved in several steps.

Let, for convenience,

$$S_n(t,q) \stackrel{\text{def}}{=} \sum_{\pi \in B_n} t^{fdes(\pi)} q^{fmaj(\pi)} \tag{13}$$

for all $n \in \mathbf{P}$, and set $S_0(t,q) \stackrel{\text{def}}{=} 1$. For example, $S_1(t,q) = 1 + tq$ and $S_2(t,q) = 1 + 2tq + tq^2 + t^2q^2 + 2t^2q^3 + t^3q^4$.

Our first result gives a recursion satisfied by the polynomials $S_n(t,q)$.

Theorem 4.3 *Let $n \in \mathbf{P}$. Then*

$$S_n(t,q) = (1 + tq + t^2q^2[2n-2]_q)S_{n-1}(t,q) + tq(1-t)(1+tq)\delta_t(S_{n-1}(t,q)).$$

Proof. The result is clear for $n \leq 2$, so fix $n \geq 3$. For $\sigma \in B_{n-1}$ and $i \in [n]$ let

$$\sigma_i \stackrel{\text{def}}{=} [\sigma(1), \ldots, \sigma(i-1), n, \sigma(i), \ldots, \sigma(n-1)]$$

and

$$\sigma_{-i} \stackrel{\text{def}}{=} [\sigma(1), \ldots, \sigma(i-1), -n, \sigma(i), \ldots, \sigma(n-1)].$$

Then clearly

$$S_n(t,q) = \sum_{\sigma \in B_{n-1}} \sum_{i=1}^{n} \left[t^{fdes(\sigma_i)} \, q^{fmaj(\sigma_i)} + t^{fdes(\sigma_{-i})} \, q^{fmaj(\sigma_{-i})} \right]. \tag{14}$$

Using (12) and our definitions it is not hard to check (though with some patience) that, for all $\sigma \in B_{n-1}$,

$$fdes(\sigma_1) = fdes(\sigma) + 1 + \chi[\sigma(1) > 0];$$

$$fdes(\sigma_{-1}) = fdes(\sigma) + \chi[\sigma(1) > 0];$$

$$fdes(\sigma_{\pm i}) = fdes(\sigma) + 2\chi[\sigma(i-1) < \sigma(i)]$$

for $i = 2, \ldots, n-1$;

$$fdes(\sigma_n) = fdes(\sigma);$$

and

$$fdes(\sigma_{-n}) = fdes(\sigma) + 2.$$

Similarly, let

$$d_i(\sigma) \stackrel{\text{def}}{=} |\{j \in Des_A(\sigma) : \; j \geq i\}| \qquad (i = 1, \ldots, n-1)$$

be the number of descents in σ from position i on. Then

$$maj_A(\sigma_i) = maj_A(\sigma) + d_i(\sigma) + (i-1)\ \chi[\sigma(i-1) < \sigma(i)] + 1$$

and

$$maj_A(\sigma_{-i}) = maj_A(\sigma) + d_i(\sigma) + (i-1)\ \chi[\sigma(i-1) < \sigma(i)]$$

for $i = 1, \ldots, n-1$;

$$maj_A(\sigma_n) = maj_A(\sigma),$$

and

$$maj_A(\sigma_{-n}) = maj_A(\sigma) + n - 1.$$

Therefore, using Theorem 4.1,

$$\begin{aligned}
&\sum_{i=1}^{n} \left[t^{fdes(\sigma_i)}\, q^{fmaj(\sigma_i)} + t^{fdes(\sigma_{-i})}\, q^{fmaj(\sigma_{-i})} \right] = \\
&\qquad = \quad t^{fdes(\sigma)+\chi[\sigma(1)>0]}\, q^{fmaj(\sigma)+2d_1(\sigma)+1}(tq+1) + \\
&\qquad + \sum_{\{i\in[2,n-1]:\ \sigma(i-1)>\sigma(i)\}} t^{fdes(\sigma)}\, q^{fmaj(\sigma)+2d_i(\sigma)+1}(q+1) + \\
&\qquad + \sum_{\{i\in[2,n-1]:\ \sigma(i-1)<\sigma(i)\}} t^{fdes(\sigma)+2}\, q^{fmaj(\sigma)+2d_i(\sigma)+2i-1}(q+1) + \\
&\qquad + \quad t^{fdes(\sigma)}\, q^{fmaj(\sigma)}(1 + t^2 q^{2n-1}).
\end{aligned} \tag{15}$$

From the definition of $d_i(\sigma)$ it is clear that

$$\sum_{\{i\in[2,n-1]:\ \sigma(i-1)>\sigma(i)\}} q^{2d_i(\sigma)} = \sum_{k=1}^{des_A(\sigma)} q^{2(k-1)} \tag{16}$$

for all $\sigma \in B_{n-1}$. On the other hand, let

$$\{i_1, i_2, \ldots, i_a\}_< \stackrel{\text{def}}{=} \{i \in [2, n-1] :\ \sigma(i-1) < \sigma(i)\},$$

so that $a = n - 2 - des_A(\sigma)$. Then

$$\sigma(i_a) > \sigma(i_a + 1) > \ldots > \sigma(n-1),$$

and therefore

$$i_a + d_{i_a}(\sigma) = i_a + (n - 1 - i_a) = n - 1.$$

Similarly, for each $j \in [a-1]$ one has

$$\sigma(i_j) > \sigma(i_j + 1) > \ldots > \sigma(i_{j+1} - 1) < \sigma(i_{j+1}),$$

and therefore

$$i_j + d_{i_j}(\sigma) = i_j + d_{i_{j+1}}(\sigma) + (i_{j+1} - i_j - 1) = i_{j+1} + d_{i_{j+1}}(\sigma) - 1.$$

This shows that

$$\sum_{\{i \in [2,n-1]:\, \sigma(i-1)<\sigma(i)\}} q^{2(d_i(\sigma)+i)} = \sum_{k=1}^{a} q^{2(n-k)} \tag{17}$$

for all $\sigma \in B_{n-1}$.

From (15), (16), and (17) we conclude that

$$\begin{aligned}
\sum_{i=1}^{n} \Big[t^{fdes(\sigma_i)}\, q^{fmaj(\sigma_i)} + t^{fdes(\sigma_{-i})}\, q^{fmaj(\sigma_{-i})} \Big] = & \\
= \quad & t^{fdes(\sigma)+\chi[\sigma(1)>0]}\, q^{fmaj(\sigma)+2des_A(\sigma)+1}(1+tq) + \\
+ \quad & t^{fdes(\sigma)} q^{fmaj(\sigma)+1}(1+q) \sum_{k=1}^{des_A(\sigma)} q^{2(k-1)} + \\
+ \quad & t^{fdes(\sigma)+2} q^{fmaj(\sigma)-1}(1+q) \sum_{k=1}^{a} q^{2(n-k)} + \\
+ \quad & t^{fdes(\sigma)} q^{fmaj(\sigma)}(1 + t^2 q^{2n-1}) = \\
= \quad & t^{fdes(\sigma)} q^{fmaj(\sigma)} \Big\{ t^{\chi[\sigma(1)>0]} q^{2des_A(\sigma)+1}(1+tq) + \\
+ \quad & \sum_{j=1}^{2des_A(\sigma)} q^j + t^2 \sum_{j=2n-2a-1}^{2n-2} q^j + (1+t^2q^{2n-1}) \Big\} = \\
= \quad & t^{fdes(\sigma)} q^{fmaj(\sigma)} \Big\{ t^{\chi[\sigma(1)>0]} q^{2des_A(\sigma)+1}(1+tq) + \\
+ \quad & \sum_{j=0}^{2des_A(\sigma)} q^j + t^2 \sum_{j=2n-2a-1}^{2n-1} q^j \Big\}.
\end{aligned}$$

Using the value of a, (10) and some case-by-case analysis (depending on the sign of $\sigma(1)$) we get

$$\begin{aligned}
\sum_{i=1}^{n} \Big[t^{fdes(\sigma_i)}\, q^{fmaj(\sigma_i)} + t^{fdes(\sigma_{-i})}\, q^{fmaj(\sigma_{-i})} \Big] = & \qquad (18) \\
= \quad & t^{fdes(\sigma)} q^{fmaj(\sigma)} \Big\{ [fdes(\sigma)+1]_q + tq^{fdes(\sigma)+1} + \\
+ \quad & t^2([2n]_q - [fdes(\sigma)+2]_q) \Big\} = \\
= \quad & t^{fdes(\sigma)} q^{fmaj(\sigma)} \{ 1 + q[fdes(\sigma)]_q + tq + tq(q-1)[fdes(\sigma)]_q + \\
+ \quad & t^2q^2([2n-2]_q - [fdes(\sigma)]_q) \Big\}. \qquad (19)
\end{aligned}$$

Substituting (19) into (14) we now obtain

$$\begin{aligned}
S_n(t,q) \quad = \quad & (1 + tq + t^2q^2[2n-2]_q) S_{n-1}(t,q) + \\
+ \quad & (q + tq(q-1) - t^2q^2) \sum_{\sigma \in B_{n-1}} [fdes(\sigma)]_q\, t^{fdes(\sigma)} q^{fmaj(\sigma)},
\end{aligned}$$

and the result follows from (13) and (6). □

Using the previous theorem we can now prove the following result.

Theorem 4.4 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\pi \in B_n} t^{fdes(\pi)} q^{fmaj(\pi)} = \prod_{i=1}^{n} (1 + tq^i) \cdot \sum_{\sigma \in S_n} t^{des_A(\sigma)} q^{maj_A(\sigma)}.$$

Proof. We show that both sides satisfy the same recursion; equality of initial conditions is easily checked ($n \leq 2$). Let, for convenience,

$$\tilde{S}_n(t,q) \stackrel{\text{def}}{=} \prod_{i=1}^{n} (1 + tq^i) \cdot A_n(t,q)$$

be the right-hand side of the formula in Theorem 4.4.

It is easily verified that

$$\delta_t \left(\prod_{i=1}^{n} (1 + tq^i) \right) = q[n]_q \prod_{i=2}^{n} (1 + tq^i),$$

so that by (7)

$$(1 + tq)\delta_t(\tilde{S}_n(t,q)) = q[n]_q \tilde{S}_n(t,q) + \prod_{i=1}^{n+1} (1 + tq^i) \cdot \delta_t(A_n(t,q)).$$

By Theorem 2.1 we therefore conclude that

$$\begin{aligned}
\tilde{S}_n(t,q) &= \prod_{i=1}^{n} (1 + tq^i) \cdot A_n(t,q) = \\
&= \prod_{i=1}^{n} (1 + tq^i) \cdot \{(1 + tq[n-1]_q) A_{n-1}(t,q) + tq(1-t)\delta_t(A_{n-1}(t,q))\} = \\
&= (1 + tq^n)(1 + tq[n-1]_q)\tilde{S}_{n-1}(t,q) + \\
&+ tq(1-t)\{(1+tq)\delta_t(\tilde{S}_{n-1}(t,q)) - q[n-1]_q \tilde{S}_{n-1}(t,q)\} = \\
&= (1 + tq + t^2q^2[2n-2]_q)\tilde{S}_{n-1}(t,q) + tq(1-t)(1+tq)\delta_t(\tilde{S}_{n-1}(t,q)),
\end{aligned}$$

and comparison with Theorem 4.3 completes the proof. □

Theorem 4.4 has several interesting consequences. The first one is that the descent statistic *fdes* yields a solution to Foata's problem (Theorem 4.2).

Proof of Theorem 4.2. This follows immediately from Theorems 4.4 and 2.2, given the definition of $A_n(t,q)$. □

This in turn implies, together with Theorem 3.2, that the two pairs of statistics $(fdes, fmaj)$ and $(ndes, nmaj)$ are equidistributed in B_n.

Corollary 4.5 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\pi \in B_n} t^{ndes(\pi)} q^{nmaj(\pi)} = \sum_{\pi \in B_n} t^{fdes(\pi)} q^{fmaj(\pi)}.$$

It would be interesting to have a direct combinatorial (i.e., bijective) proof of this result.

Finally, the special case $t = 1$ of Corollary 4.5, together with Proposition 3.1, implies the following result, which extends Theorem 2.2 of [1].

Corollary 4.6 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\pi \in B_n} q^{l(\pi)} = \sum_{\pi \in B_n} q^{nmaj(\pi)} = \sum_{\pi \in B_n} q^{fmaj(\pi)}.$$

Acknowledgments. This paper grew out of stimulating discussions with Dominique Foata, Ira Gessel and Doron Zeilberger during the conference "Classical Combinatorics" in honor of Foata's 65^{th} birthday. The work on it was carried out while the second author was visiting the other authors at Bar-Ilan University.

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